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Method of Orthogonal Potentials Developed for the Analysis of TEM Mode Electromagnetic Resonators

Introduction

This note develops a technique that simplifies specifying the boundary conditions for 3D problems in the FEA analysis program “FlexPDE” from PDE Solutions (www.pdesolutions.com). I have been using this program almost from the time it was released under a different name by a different company. The program began as a 2D scalar solver and has evolved a lot in the years with a 3D version now available. A user completely describes the problem equations, boundaries, and boundary conditions in the input text file. As a result, the program is not limited to any pre-programmed problem set as is true with many other programs. Additionally, it completely automates the finite element mesh generation and refinement process.

The program applies scalar boundary conditions consisting of both “Value” boundary conditions and so called “Natural” boundary conditions. The natural boundary conditions are referred to as “insulating conditions” and their effect is based on the particular form of the user equations. Since this is a scalar solver, vector based boundary equations are not straightforward unless they fall on constant coordinate surfaces. A combination of these effects has led me to develop a method of solving for the resonant frequency and fields within a 3D TEM mode (transmission line mode) resonator that simplifies and standardizes the problem of specifying the boundary conditions.

FlexPDE applies integration-by-parts to all terms of the user specified partial differential equation to be solved that contain *second-order derivatives* of the system variables. As far as electromagnetic fields are concerned, this basically means applying either Stokes’s theorem or Gauss’s law to specify the fields on the boundaries. The Natural boundary condition (BC) specifies the resulting integrand. In the following examples, A is a vector field, u is a scalar field and n is a unit normal to the enclosing surface or surrounding boundary.

$$\int_V (\nabla \times A) dV = \int_S (n \times A) dS \quad \text{Natural BC = the value of } (n \times A) \text{ on the surface}$$

$$\int_S (\nabla \times A) dS = \int_l (n \times A) dl \quad \text{Natural BC = the value of } (n \times A) \text{ on the boundary}$$

$$\int_V (\nabla \bullet A) dV = \int_l (n \bullet A) dS \quad \text{Natural BC = the value of } (n \bullet A) \text{ on the surface}$$

$$\int_S (\nabla \bullet A) dS = \int_l (n \bullet A) dl \quad \text{Natural BC = the value of } (n \bullet A) \text{ on the boundary}$$

$$\int_V (\nabla \times \nabla \times A) dV = \int_S (n \times \nabla \times A) dS \quad \text{Natural BC = the value of } (n \times \nabla \times A) \text{ on the surface}$$

$$\int_V (\nabla \bullet \nabla u) dV = \int_l (n \bullet \nabla u) dS \quad \text{Natural BC = the value of } (n \bullet \nabla u) \text{ on the surface}$$

Development

In a TEM mode resonator, no field components are directed along the wave path. The TEM condition is defined by fields that obey:

$$\nabla_t \times E_t = 0 \quad \text{and} \quad \nabla_t \times H_t = 0$$

In the above relationships, the "t" subscript specifies the components and operators existing or acting transverse to the wave direction. For simple problems, such as those found in most textbooks, one can take the wave direction to be uniformly along one axis normally taken to be the z axis. The technique to be developed here is not limited to wave propagation along one or more coordinate axis's. Another consequence of the above relationships is that the transverse fields may be described by the transverse gradient of a scalar field.

$$E_t = -\nabla_t V \quad \text{and} \quad H_t = -\nabla_t V_m$$

At this point, it is useful to review the form of the standing wave pattern for the potential field along a ¼ wave resonant coaxial transmission line with outer radius "b" and inner radius "a".

$$V(r,z) = V_o \frac{\left(\ln \frac{b}{r}\right)}{\left(\ln \frac{b}{a}\right)} \cos(Kz)$$

The basic form of the above equation is found in many resonant systems and may be cast into the more general form

$$V = V_t V_p \,,$$

where in this case

$$V_t \equiv V_o \frac{\left(\ln \frac{b}{r}\right)}{\left(\ln \frac{b}{a}\right)} \quad \text{and} \quad V_p \equiv \cos(Kz).$$

The subscript "p" is taken to mean parallel to the wave direction and as before "t" means transverse to the wave direction. With these ideas in mind, the technique can be developed.

Technique: Solve for a scalar field (V) that can be resolved into a scalar field (V_t) representing the transverse field components that obey the electrostatic field solution multiplying a scalar field (V_p) that represents the field behavior along the wave direction.

The scalar field must obey the Helmholtz equation;

$$\nabla^2 V + K^2 V = 0$$

$$\Rightarrow \nabla^2 V_t V_p + K^2 V_t V_p = 0$$

$$\Rightarrow V_p \nabla^2 V_t + V_t \nabla^2 V_p + 2\left(\nabla V_p \bullet \nabla V_t\right) + K^2 V_t V_p = 0$$

Since V_p only varies along the wave direction and V_t only varies transverse to the wave direction (i.e. $\nabla_t V_p = \nabla_p V_t = 0$) then the term $2\left(\nabla V_p \bullet \nabla V_t\right) = 0$. Additionally, V_t describes a field identical to a charge free electrostatic field that is known to obey Laplace's equation $\nabla^2 V_t = 0$. Using this information the equation may be separated into

$$\nabla^2 V_t = 0,$$

and

$$\nabla^2 V_p + K^2 V_p = 0 \text{ with } K^2 = \omega^2 \mu\varepsilon$$

E, H Field, ω

For the Electric Field:

$$E_t = -\nabla_t V = -\nabla_t V_p V_t = -V_p \nabla_t V_t - V_t \nabla_t V_p = -V_p \nabla_t V_t$$

since $\nabla_t V_p = 0$.

Note: $\nabla V_t = \nabla_t V_t + \nabla_p V_t = \nabla_t V_t \therefore E_t = -V_p \nabla V_t$

For the Magnetic Field:

$$\nabla \times E_t = -j\omega\mu H_t$$

$$\Rightarrow H_t = \frac{\nabla \times E_t}{-j\omega\mu} = \frac{\nabla \times V_p \nabla V_t}{j\omega\mu}$$

$$\Rightarrow H_t = \frac{V_p \nabla \times \nabla V_t + \nabla V_p \times \nabla V_t}{j\omega\mu} = \frac{\nabla V_p \times \nabla V_t}{j\omega\mu}$$

$$\text{since } \nabla \times \nabla V_t = 0$$

For ω:

$$\text{Solve } \nabla^2 V_p + K^2 V_p = 0 \text{ varying “K” such that } \frac{\varepsilon}{2} \int_{Volume} \left| E_t \right|^2 dV = \frac{\mu}{2} \int_{Volume} \left| H_t \right|^2 dV$$

Summary Equations

$$V = V_t V_p$$

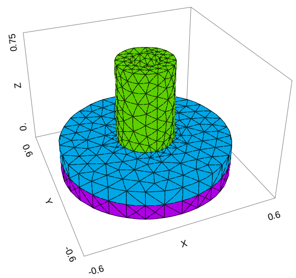
$$\nabla^2 V_t = 0$$

$$\nabla^2 V_p + K^2 V_p = 0 \text{ with } K^2 = \omega^2 \mu\varepsilon \text{ such that } \frac{\varepsilon}{2} \int_{Volume} \left| E_t \right|^2 dV = \frac{\mu}{2} \int_{Volume} \left| H_t \right|^2 dV$$

$$\text{where } E_t = -V_p \nabla V_t \text{ and } H_t = \frac{\nabla V_p \times \nabla V_t}{j\omega\mu}$$

Test Case

The following coaxial disk resonator is analyzed in 2.5D and 3D. The 2.5D analysis exploits the circularly symmetric condition using a standard technique, whereas the 3D analysis is done using the technique developed here. The resonator is assumed symmetric about the large circular end surface and shorted at the small circular end surface.



The 2.5D analysis solves the equation $\nabla \times \nabla \times H_\phi - K^2 H_\phi = 0$ with the boundary conditions VALUE(H_ϕ)=0 on the large end and NATURAL(H_ϕ)=0 on all other surfaces. The natural boundary condition in this case specifies the value of $n \times \nabla \times H_\phi$ that by Maxwell's equations is equivalent to setting the tangential components of E to 0.

The 3D case boundary conditions for V_t are VALUE(V_t)=0 on the outer boundaries and VALUE(V_t)=1 on the inner boundaries spanning from the large to small coaxial ends with both the large and small coaxial end surfaces set to NATURAL(V_t)=0. The boundary conditions for V_p are VALUE(V_p)=1 on the large coaxial end surface VALUE(V_p)=0 on the small coaxial end surface with NATURAL(V_p)=0 on all other surfaces.

Appendix I and II print the input files for these cases for more detailed information. The following table summarizes the results.

	2.5D	**3D**
F(MHz)	58.08	59.98
Q	9800	9630
% error	0	3.3

The following plots compare the results from the 2.5D to the 3D analysis. The 2.5D results do not include a potential plot because one is not possible.

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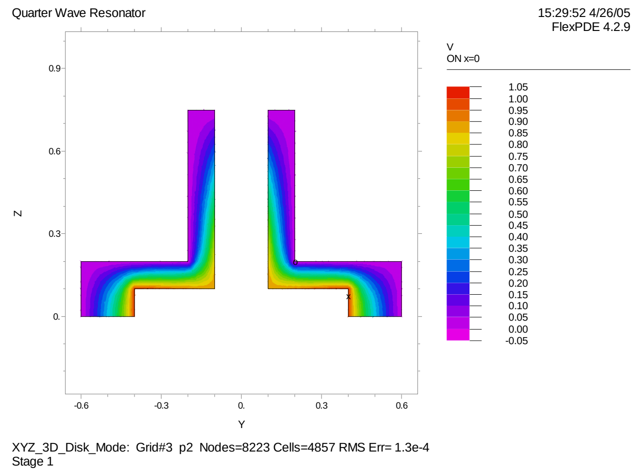
<tr><th>2.5D Blank, Grid, Hmag, Emag</th><th>3D V, Grid, Hmag, Emag</th></tr>
<tr><td></td><td>
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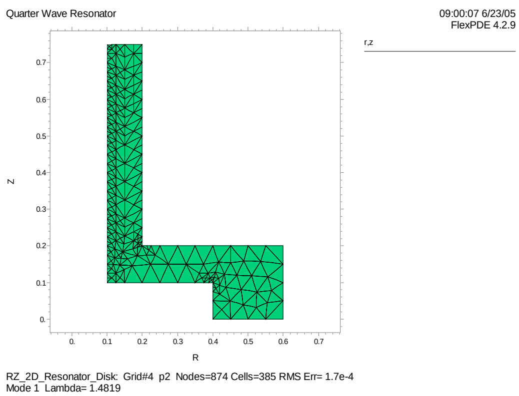
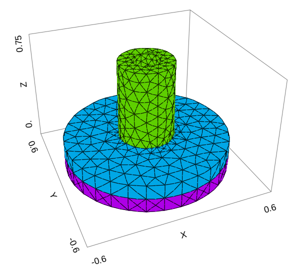
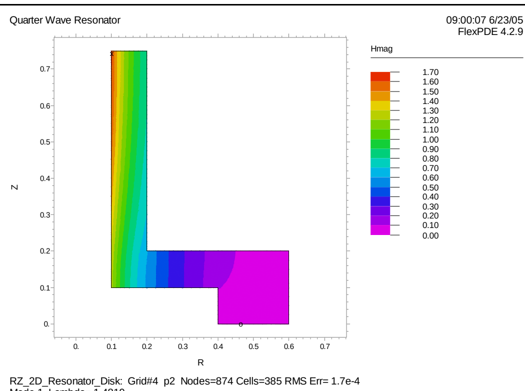
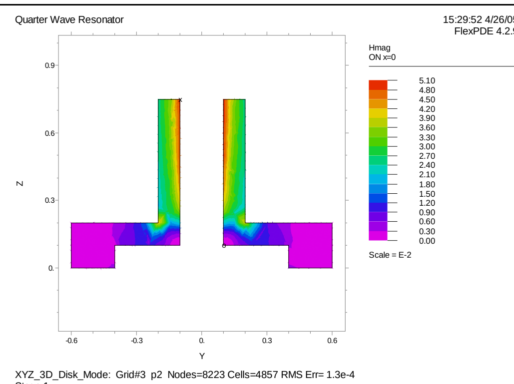
<tr><td>
</td><td></td></tr>
<tr><td>
</td><td>
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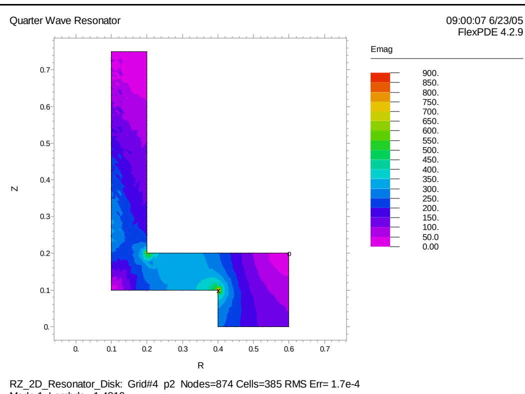
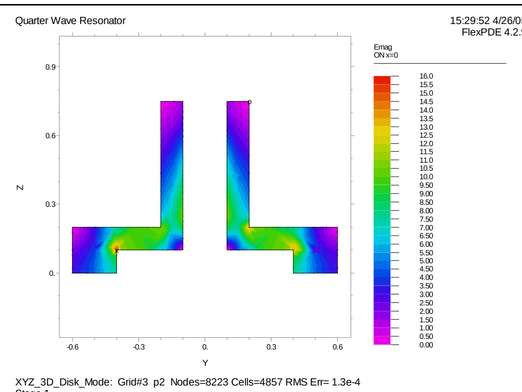
<tr><td>
</td><td>
</td></tr>
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Appendix I (The 2.5D FlexPDE input file)

```
TITLE
  '2.5D Quarter Wave Resonator'

Coordinates
  ycylinder("r","z")

SELECT
  errlim=1E-3
  modes = 2
  thermal_colors on
  plotintegrate off

VARIABLES
 Hphi

DEFINITIONS

        { Resonator Extents in Meters }

   r1= 0.1 r2=0.2 r3=0.4 r4=0.6 L1= 0.1 L2=0.2 L3=0.75

        { Material Constants }

  eps0= 8.854e-12  { Farads/m }         { Permitivity of Free Space}
  epr=1.0                               { Relative Permitivity }
  mus0=4*pi*1e-7     { Henries/m }      { Permeability of Free Space }
  mur=1.0                               { Relative Permeability }
  eps= epr*eps0                         { Resultant Permitivity }
  mus= mur*mus0                         { Resultant Permeability }
  sigma=5.8e+7 { mhos/m }               { conductivity of copper at 20 degrees C }
  Vuser=1.05 { volts peak }             { user desired peak voltage on "user" path }

        { Computed Results }

  omega=sqrt(lambda/(mus*eps))          { Angular Frequency }
  freq=omega/(2*pi)                     { Frequency }
  H=vector(0,0,Hphi)
  Hmag=Magnitude(H)
  Er=-(1/(omega*eps))*dz(Hphi)
  Ez=(1/(omega*eps))*(1/r)*dr(r*Hphi)
  E=vector(Er,Ez)
  Emag=Magnitude(E)
  RR=sqrt((mus*omega)/(2*sigma))                  { Surface Resistance }
  PC=(1/2)*RR*Sintegral(abs(Hmag)^2,'perimeter'){ Conduction Losses }
```

```
  UE=(eps/2)*integral(Emag^2,'cavity')            { Stored Electric Energy }
  UH=(mus/2)*integral(Hmag^2,'cavity')            { Stored Magnetic Energy }
  Q=(omega*UE)/PC                                 { Resonator Quality Factor }

        { User Scaling }

  V=-bintegral(tangential(E),'user')              { Voltage along user path }
  Kfactor=abs(Vuser/V)                            { user scaling factor based on Vuser
}
  PCS=PC*Kfactor^2                                { Scaled Conduction Losses }
  UES=UE*Kfactor^2                                { Scaled Stored Energy }
  UHS=UH*Kfactor^2
  CS=(2*UE)/abs(V)^2                              { User Path Shunt Capacitance }
  LS=1/(omega^2*CS)                               { User Path Shunt Inductance }
  RS=Q/(omega*CS)                                 { User Path Shunt Resistance }

        { Equations to be Solved }

EQUATIONS
 Hphi: Curl(Curl(Hphi))-lambda*Hphi=0
 ! Hphi: Div(Grad(Hphi))+lambda*Hphi=0

        { Resonator Boundaries and Boundary Conditions }

BOUNDARIES
region 1 'cavity'
  start 'perimeter' (r1, L1)
    Natural(Hphi)=0   line to (r3, L1)
                      line to (r3, 0)
    Value(Hphi)=0             line to (r4, 0)
    Natural(Hphi)=0   line to (r4, L2)
                      line to (r2, L2)
                      line to (r2, L3)
                      line to (r1, L3)
                      line to finish

        { Define the User Path }

feature 1
  start 'User' (r3, 0) line to (r4, 0)

        { Requested Outputs for each Mode }

PLOTS
   grid(r,z)
   contour(Er) painted
```

```
    contour(Ez) painted
    contour(Hmag) painted
    contour(Emag) painted
    elevation(Emag) ON 'perimeter'
    elevation(Er) from (r1,0) to (r1,L3)
    elevation(Er) on 'user'
    vector(E) norm notips

SUMMARY
   report(freq) as "frequency"
   report(lambda)
   report(UES)
   report(Q)
   report(UHS)
   report(CS)
   report(LS)
   report(PCS)
   report(UES)
   report(Vuser)
END
```

Appendix II (The 3D FlexPDE input file)

```
TITLE
  '3D analysis of a quarter wave case'

  { Comment out coordinates section for rectangular case }

Coordinates
  Cartesian3

SELECT
  errlim=5E-4
  stages=1
  thermal_colors on
  plotintegrate off

VARIABLES
  Vt
  Vp

 DEFINITIONS

        { Resonator Extents in Meters }

   r1= 0.1 r2=0.2 r3=0.4 r4=0.6 L1= 0.1 L2=0.2 L3=0.75          { Resonator
Maxumum Extents }

        { Material Constants }

  eps0= 8.854e-12  { Farads/m }        { Permitivity of Free Space}
  epr=1.0                             { Relative Permitivity }
  mus0=4*pi*1e-7     { Henries/m }    { Permeability of Free Space }
  mur=1.0                             { Relative Permeability }
  eps= epr*eps0                       { Resultant Permitivity }
  mus= mur*mus0                       { Resultant Permeability }
  eta = sqrt(mus/eps)                 { impedance of space }
  sigma=5.8e+7 { mhos/m }             { conductivity of copper at 20 degrees C }
  Vuser=100 { volts peak }            { user desired peak voltage on "user" path }

        { Computed Results }

  lambda1=1.380 + stage*0.2           { Mode 1 eigenvalue }
  !lambda1=21.54930+ stage*0.2        { Mode 2 eigenvalue }

   omega=sqrt(lambda1/(mus*eps))      { Angular Frequency }
   freq=omega/(2*pi)                  { Frequency }
```

```
  V=Vp*Vt                                          { Voltage }
  E=-Vp*Grad(Vt)-Vt*Grad(Vp)                       { Electric Field }
  H=(1/(omega*mus))*cross(grad(Vp),grad(Vt))       { Magnetic Field }
  Emag=Magnitude(E)
  Hmag=Magnitude(H)
  UE=(eps/2)*integral(Emag^2)                      { Stored Electric Energy }
  UH=(mus/2)*integral(Hmag^2)                      { Stored Magnetic Energy }
  RR=sqrt((mus*omega)/(2*sigma))                   { Surface Resistance }
  PC=(1/2)*RR*Sintegral(abs(Hmag)^2)               { Conduction Losses }
  Q=(omega*UE)/PC                                  { Resonator Quality Factor }
  error = abs(1-sqrt(UE/UH))*100                   { +/- error in resonant
frequency }

        { Equations to be Solved }

EQUATIONS
  Vp: Div(Grad(Vp)) + Lambda1*Vp=0
  Vt: Div(Grad(Vt))=0

        { Geometry }

EXTRUSION
  Surface 'Open' z=0
   Layer 'Low_Z'
  Surface 'S1' z=L1
   Layer 'High_Z'
  Surface 'S2' z=L2
   Layer 'Stem'
  Surface 'Short' z=L3

        { Resonator Boundaries and Boundary Conditions }

BOUNDARIES
   Surface 'Open' Natural(Vt)=0 Value(Vp)=1
   Surface 'Short' Natural(Vt)=0 Value(Vp)=0

region 1 'Extents'
  start 'outer' (r4, 0) Value(Vt)=0 Natural(Vp)=0
     ARC (CENTER =0,0) ANGLE=360 to finish

Limited region 'V1'
  Layer 'Low_Z' VOID
  start 'outer' (r3, 0) Value(Vt)=1 Natural(Vp)=0
     ARC (CENTER =0,0) ANGLE=360 to finish
  Surface 'S1' Value(Vt)=1 Natural(Vp)=0
```

```
Limited region 'V2'
  Layer 'High_Z' VOID
  start 'outer' (r1, 0) Value(Vt)=1 Natural(Vp)=0
     ARC (CENTER =0,0) ANGLE=360 to finish

Limited region 'V3'
  Layer 'Stem' VOID
  start 'outer' (r1, 0) Value(Vt)=1 Natural(Vp)=0
     ARC (CENTER =0,0) ANGLE=360 to finish

Limited region 'V4'
  Layer 'Stem' VOID
  start 'outer' (r4, 0)
     ARC (CENTER =0,0) ANGLE=360 to finish
  start 'inner' (r2, 0) Value(Vt)=0 Natural(Vp)=0
      ARC (CENTER =0,0) ANGLE=360 to finish
  Surface 'S2' Value(Vt)=0 Natural(Vp)=0

        { Requested Outputs for each Mode }

PLOTS
  contour(Vp) ON x=0 painted
  contour(Vt) ON x=0 painted
  contour(V) ON x=0 painted
  contour(Hmag) ON x=0 painted
  contour(Emag) ON x=0 painted

SUMMARY

  report(freq) as "frequency"
  report(PC)
  report(Q)
  report(error)
  report(UE)
  report(UH)

END
```